It's a wonderful line

Jose Tristan Reyes
De La Salle University
jose.tristan.reyes@dlsu.edu.ph

Abstract: Assuming that time is a continuum, we explore a mathematically plausible model for the literary device of following an alternate timeline. We take the real line as a model for the progression of events. We show that the real line is a continuum that contains subsets that are themselves continua. We also assume that our perception of time is segmented. These assumptions allow us to pose two mathematical models that explain how an alternative timeline is created.

Key Words: real line; completeness axiom; continuum

1. INTRODUCTION

In Frank Capra's It's a Wonderful Life, James Stewart's character, George Bailey, in a moment of despair wishes he'd never been born. Immediately after, his guardian angel, Clarence, proceeds to show him what would have happened if, indeed, he'd never been born. The introduction of an alternate timeline is a familiar enough literary device. Is there a mathematical theory which could explain or at least plausibly model this phenomenon?

To address this, we make a commonly held assumption. We assume that time flows linearly in a continuum. In which case, we will take time as being modeled by the real line.

First, we examine a select set of properties of the real line and we use these properties to construct our model.

2. THE REAL LINE

One of the fundamental assumptions about the structure of the real line is the Completeness Axiom.

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Let S be a nonempty set of real numbers which is bounded above; that is, there exists a number M such that $s \leq M$ for all $s \in S$. Then S has a least upper bound. ■

One consequence of the Completeness Axiom is that the real line has no holes. That is to say that the real line is a continuum. First, we take a look at one formulation of the Axiom of Archimedes.

Proposition 1. (Axiom of Archimedes)
Given an arbitrary positive number ϵ, there exists a natural number N such that $\frac{1}{N} < \epsilon$.

Proof: We assume that $\epsilon < 1$ is arbitrarily small, for if say $\epsilon \geq 1$, then we observe that $\frac{1}{2} < \epsilon$ and we may take $N = 2$.

Consider the set $A = \left\{n \in \mathbb{N} : \frac{1}{n} \geq \epsilon\right\}$. From above, $A \neq \emptyset$ since $1 \in A$. Further, A is bounded above by $\frac{1}{\epsilon}$. Applying the Completeness Axiom, we take μ as the least upper bound for A. It follows that $\mu - \epsilon$ is not an upper bound for A. Hence, we can find some integer $m \in A$ such that $m > \mu - \epsilon$. Observe that $m + 1 > \mu - \epsilon + 1 > \mu$. Therefore $m + 1 \notin A$. That is $\frac{1}{m+1} < \epsilon$. ■

Before proceeding, we introduce a notational convention to distinguish between the representation of a point in $\mathbb{R}^2$ and a finite open interval in $\mathbb{R}$. We retain the use of the notation (a, b) to denote a point whose coordinates are $x = a$ and $y = b$. For the open interval consisting of all real numbers x such that $a < x < b$, we use the notation $I(a, b)$.

Proposition 2. The real line is a continuum.

Proof:

It would be enough to show that between any two real numbers is another real number.

Take $a, b \in \mathbb{R}$, with $a < b$. Let $\epsilon = b - a$. From Proposition 1, take $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. We then have $a < a + \frac{1}{n} < a + \epsilon = a + (b - a) = b$. Thus, we have produced a number which lies between a and b.

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The proof of Proposition 2 naturally leads us to conclude that any interval is also a continuum.

Corollary 3. For $a, b \in \mathbb{R}$, $a < b$, the interval $I(a, b)$ is a continuum.

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At this point, we address whether one continuum is essentially the same as any other continuum. Specifically, we determine whether one interval could replace any other interval. After all, one clearly sees that an interval of length 1, say $I(0, 1)$ does not have the same length as the interval $I(0, 2)$.

It is worth noting that when dealing with the infinite, our intuition sometimes fails us. Hence, we need to be more precise when we discuss specific concepts, such as when one interval could take the place of another. Since we will relate this discussion with traversing a timeline, the question of replaceability of intervals will be viewed from the perspective or having exactly the same number of time points. Thus, our question translates to determining whether we could place one interval in a one-to-one correspondence with another interval.

For reasons outside the scope of what we need, the answer is no, if one interval does not include the same number of endpoints as the other, say one is the open interval $I(a, b)$ while the other is the closed interval $[c, d]$. Hence, we will focus on open intervals. Another reason for dealing with open intervals is because of our understanding of time. We do not really know when time began, nor are we aware of when it will end. Hence, our grasp of the timeline is indeed modeled by our intuitive grasp of the real line

It is straightforward to show that one open interval can be mapped bijectively onto another open interval. Consider the open intervals $I(a, b)$ and $I(c, d)$. We may visualize $I(a, b)$ as a subset of the $x-$axis while $I(c, d)$ is a subset of the $y-$axis. The correspondence is then given by the equation of the line joining the points (a, c) and (b, d), namely

$$f(x) = \frac{d-c}{b-a}(x - a) + c. \quad (1)$$

Figure 1 below illustrates the correspondence. Note that the slope of the line introduces a scaling factor that allows us to map intervals of differing lengths.

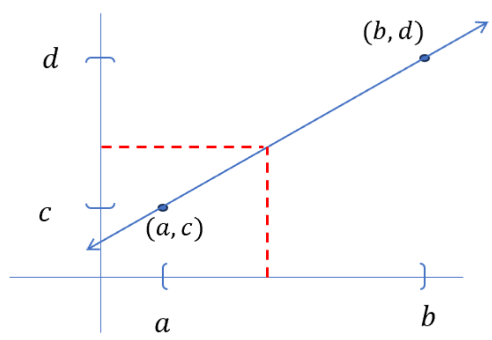

Fig. 1. Correspondence between (a, b) and (c, d).

In the above discussion, we could take $a = 0$ and $b = 1$ to get a more explicit result.

Proposition 4. Any open interval may be put in a one-to-one correspondence with the open interval $I(0,1)$.

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Next, we note that the real line is in fact an open interval, even though it is an infinite one. Is it possible to put the real line into a one-to-one correspondence with a finite interval, say $I(a, b)$? Intuitively, one may think that this is not possible since $\mathbb{R}$ is infinitely long while $I(a, b)$ isn't. However, a quick look at elementary functions tells us that the correspondence is possible. Consider the restricted tangent function, $f(x) = \tan x$, where we restrict the domain to the set $x \in I\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Figure 2 illustrates the correspondence between $I\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mathbb{R}$.

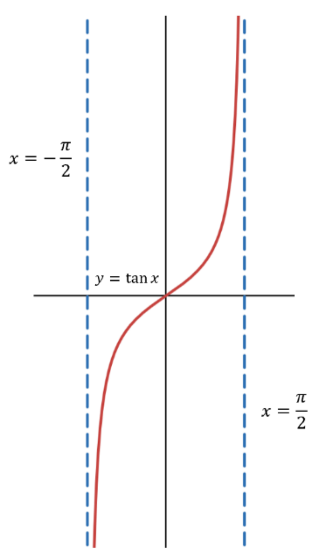

Fig. 2. $f(x) = \tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

A straightforward application of (1) allows us to define a correspondence between the interval $I(0,1)$ and the interval $I\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Taking the composition with the function $f(x) = \tan\left(\frac{\pi x}{2}\right)$, where $x \in I(-1, 1)$, we get a correspondence between $I(0,1)$ and the real line. Specifically, we may take $f(x) = \tan\left[\pi\left(x - \frac{1}{2}\right)\right]$ to define a bijection from $I(0,1)$ onto $\mathbb{R}$. We have just shown that

Proposition 5. Any open interval may be put in a one-to-one correspondence with the real line.

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3. THE MODEL

3.1 Basic assumptions

We reiterate the assumption that time flows along in a linear fashion and may thus be associated with the real line. Consequently, we take time as a linear continuum.

We make the added assumption that while we experience time as a continuum, we do not perceive it as such. In a sense, a singular point in time is elusive. One is hard-pressed to describe the experience of precisely one instant of time. Being in a continuum, one time point flows into the next. In this elusiveness, it can be argued that we experience time in segments. When we blink, the moment our mind wanders, we have a span of time for which we have no conscious recollection. It may be an infinitesimal span but it has a nonzero duration nonetheless.

While time flows in a continuum, our perception of it is imperfect. The moment one starts to consciously grasp a single time point, it has already flowed to the neighboring time point.

3.2 A primitive model

Here, we focus on time as viewed by a specific individual. In this model, there are as many timelines as there are individuals. Each individual exists on his/her own timeline. As mentioned in the earlier section, the view for each individual is imperfect and is segmented. For a chosen individual, during any one of the moments of inattention, a duration of time elapses. During this arbitrary interval, there are as manty time points as there are in our current timeline. Hence, a new set of events may transpire distinct from the events the individual's mind consciously follows.

In the individual's current time view, there is a subinterval of which he is oblivious, which carries with it its own set of events. These are the events that transpire in an alternate timeline. The infinitesimal duration is the alternate timeline. One cannot argue that the interval is too short to contain a life event since Proposition 4 tells us that any interval has the same number of points as any other interval.

This primitive model, however, does not exactly address George Bailey's situation. In Bailey's case, he is no longer a part of the alternative time line. We consider a modified version of this primitive model.

3.3 Branching primitive intervals

Here, we consider a universal time line, which we will denote by T, along which exists one generated universe. Since the universe exists on one timeline, each individual, through some agency we assume exists, may view portions of the timeline. That is, the events in the timeline are assumed to be discoverable by those that exist on the timeline. While there is only one timeline, an individual in this universe may generate branches emanating from the universal timeline. The created branches follow the primitive model.

The generated primitive models are still part of the universal timeline and are thus discoverable by anyone in the universe. The inclusion of the primitive branches may be shown to be included in the universal timeline T since each generated subinterval may be embedded in T using Proposition 4; that is, by mapping one open interval into another open interval. Alternatively, one may embed T onto the generated subinterval of time by using Proposition 5.

One may view the configuration as a line along which branches sprout, which in turn sprout their own branches. The universal timeline T is maintained since an individual may experience only one timeline at any given time, effectively hiding all branches for that individual while he/she pursues a specific set of branches.

3. SUMMARY AND CONCLUSIONS

Inquiring about a mathematical model that may help establish the matter of an alternate timeline, the common assumption that time flows continuously in a linear fashion allowed us to model time using the real line. A key property of $\mathbb{R}$ is that every open interval is itself a continuum which may be mapped to any other open interval, even onto $\mathbb{R}$ itself. This allowed the system of taking a finite segment of the timeline to be a full timeline itself. In more descriptive terms, one may say that we are able to live a lifetime in the blink of an eye.

A primitive model is described where one simply takes an infinitesimal subset of time to host an alternative set of events. This primitive model is used to create a slightly more sophisticated model involving a universal timeline with branching primitive models.

The models presented rely on the claimed physical impossibility of perfectly appreciating time. Would a perfect perception of time simplify or complicate our model? This is more a philosophical question rather than a mathematical one.

4. REFERENCES

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